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COMPOSITE DESIGN SYNTHESIS

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REPORT SUMMARY

This is the first semi-annual technical report describing the work done and the accomplishments of the research effort for ONR-ARPA titled, "Composite Design Synthesis".

The research being carried out deals with the problem of design synthesis in heterogeneous elasticity. Design synthesis is defined as the achievement of a desired design criterion, i.e., stress distribution, strength-to-weight ratio, etc., by preselecting a stress or displacement pattern in a stretched plate and then determining the variation of the elastic moduli that is required to permit the desired effects. The accomplishment of the desired effects requires the solution of the governing equations of elasticity, particularly the compatibility equation, in terms of preselected stress fields in the body for unknown material properties which are spatial functions.

Since the initiation of the program, progress has been made in several areas. The compatibility equation for generalized plane stress has been generated in terms of both rectangular and polar coordinates. Solutions for material property variation for the pressurized annular disk have been generated wherein the stress condition imposed was one of constant circumferential stress through the wall of the disk. This solution showed that such a stress distribution can be achieved if a smoothly varying modulus of elasticity is developed such that it is a given function of the radius. Material property relations of a modified type of orthotropy as well as isotropy were employed and curves plotted for a range of material properties. It was shown that quite different distributions of the moduli are required for the externally pressurized disk as compared to the internally pressurized disk. These moduli functions were derived directly from the compatibility equation. The work done showed that it is theoretically possible to achieve a maximum circumferential stress in a pressurized disk that is considerably lower than can be achieved in the homogeneous isotropic or orthotropic case, and in fact, for the internally pressurized disk this maximum stress can be considerably lower than the applied pressure.

The governing equations of the heterogeneous rotating, uniform thickness, rotating annular disk have been formulated and the solutions are being determined numerically. In this case the material density as well as

the moduli can be considered as spatial functions, thus introducing another degree of freedom that might be employed by a designer.

A stress function has been developed which if implemented by the correct material property variation will eliminate the stress concentration found around small holes in large plates. This particular application has great usefulness in the fabrication of structures that are secured by rivets or for structures which must contain circular cutouts.

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SYMBOLS

Symbol	Quantity
a, b	inside and outside radius
a_{ij}	material coefficients
c	dimensionless shear-modulus coefficient $\left(\equiv \frac{E_\theta}{2G}\right)$
A_o, B_o, C_o, C_1, C_2	constants of integration
e	orthotropic ratio $\left(\equiv \frac{E_\theta}{E_r}\right)$
E_i	modulus of elasticity corresponding to the subscripted direction $(i = r, \theta)$
f	$\sqrt{e}$
$f_1(\theta), f_2(\theta)$	functions of θ only
G	modulus of rigidity $\left(\equiv G_{r\theta} \equiv \frac{1}{a_{66}}\right)$
g	acceleration of gravity
k	radius ratio $(\equiv b/a)$
k_1	orthotropic ratio $(\equiv e)$
k_2	Poisson's ratio in tangential direction $(\equiv \nu_{\theta r})$
m	$\cos \phi$
n	$\sin \phi$
P	internal pressure
q	external pressure
r	radius
T	temperature difference function
u, v	displacements
R, X, Y	body forces

Symbol	Quantity
α	coefficient of linear thermal expansion
β	exponent $(\equiv -k_1/k_2)$
γ	material density
ϵ	strain component with one or two subscripts
Θ	body force in tangential direction
θ	angular position
λ	exponent
ν	Poisson's ratio in tangential direction $(\equiv \nu_{\theta r})$
ξ	exponent $\left(\equiv - \left[\frac{k_1 - k_2}{(1-k_2)k_2}\right]\right)$
ρ	dimensionless position ratio $(\equiv r/b)$
ϕ	coordinate system offset angle
Ψ	stress function
ω	rotational velocity

INTRODUCTION

It has long been recognized that structural elements composed of composite materials, such as glass, boron, carbon, or other filaments, embedded in a suitable matrix, such as epoxy or polyester, offer outstanding strength-to-weight ratios. The potential of such materials is considered so great that material scientists and engineers believe that they will form the bulk of the structural materials of the future. Though the application of such materials has been a growing part of the state of the art for structural components, particularly in the aerospace industry, the translation of the concept of fibrous composites into a primary load carrying structure has been and remains a challenging process.

The present and growing use of structural elements fabricated from composite materials creates the need for the development of a rational analytic design basis, which to a great extent is presently non-existent. This is not to be construed as meaning that little or no research on composite materials has been carried out. On the contrary, a large amount of literature has been generated dealing with both the determination of the mechanical properties of these materials and the analysis of specific structures fabricated from them.

In general, from the microscopic viewpoint, research on the mechanical properties of composites has dealt with the determination of such properties for materials having given component elements ordered in fixed spatial relationships. The spatial relation in these cases might have a high degree of symmetry, as in long or continuous filament composites, or a completely random or homogeneously disordered array as usually employed in short carbon or boron fiber composites. Such research has been directed towards the creation of analytic or experimental methods of determining the mechanical properties of composite structures in terms of the known properties of the composites' components, characteristic of this approach are the works of Sayers and Hanley [1]*, Chen and Cheng [2], Hill [3], and Gaonkar [4], among others.

In the analysis of structures composed of such composites, the material has generally been treated as exhibiting gross, homogeneous,

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*Numbers in brackets are references found at the end of this report.

isotropic or anisotropic mechanical properties. These gross properties, when entered into the constitutive equations defining the material, have allowed analyses of such structures through the classical methods of the theory of elasticity, plates and shells, vibration, and others. Along these lines, the concept of the "unidirectional lamina" was introduced and utilized as the "fundamental unit of material" in design and analysis. This procedure employs test data obtained from a unidirectional lamina as the basis for the design of laminated components and structures. Exemplary work done along these lines has been carried out by Dong [5], Tsai [6,7], Tsai and Azzi [8], Whitney [9], and Whitney and Liessa [10,11], also among others.

All of these analyses have dealt with materials and structures that have predetermined mechanical and behavioral characteristics. That is, once the geometric array and the constituents of the composite are prescribed, then the mechanical properties of the material and the response characteristics of a structure fabricated from such a material have been inherently established. Analysis will merely determine what these properties and response characteristics are.

When dealing with composite materials, the analytical procedures discussed above appear to be highly inefficient in many applications. The designer of fibrous composite structures is presented with numerous degrees of freedom and an opportunity to exercise ingenuity totally unavailable to him with conventional materials. Composites, whether filamentary, fibrous, or sintered or fused metallics, are capable of being tailored to meet specific requirements. When considering specific structural applications for such materials, it would be logical to assume that a structure could be optimized, depending, of course, upon the optimization criterion, by varying the mechanical properties of the material throughout the structure. Further it would be logical to bypass analysis completely and define this now nonhomogeneous structure by some means of design synthesis. Admittedly, the creation of a design synthesis procedure to adequately handle most problems in structural design is quite difficult. However, the concept of design synthesis to determine the variation of the mechanical properties of a material within a structure so as to achieve a desired stress or deformation pattern in that structure is one capable of being developed.

The work described herein deals with the first phase of an effort to develop a design synthesis methodology for composites. It is directed specifically to the problem of heterogeneous plane elasticity.

GENERAL DISCUSSION

Consider the following question:

Given a plane elastic body with known boundary tractions and/or displacements, can the mechanical properties of the continuum be described such that an "arbitrary" stress distribution within the body is met?

The term "arbitrary" is to be understood as defining a family of stress distributions that are preselected but still conform to equilibrium requirements and boundary conditions. To answer this question we start by making the following two basic assumptions:

(1) the classical equations of linear elasticity are valid in this application, and

(2) the mechanical properties of the continuum can be expressed as spatial functions.

Following from these assumptions the well known governing relations of generalized plane stress, given in rectangular coordinates are as follows:

Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \,, \tag{1}$$

Strain-Displacement Equations

$$\epsilon_x = \frac{\partial u}{\partial x} \,, \quad \epsilon_y = \frac{\partial v}{\partial y} \,,$$

$$\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \,, \tag{2}$$

Compatibility Equation

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \,. \tag{3}$$

Assuming that the continuum exhibits orthotropic material properties and neglecting time and strain rate effects, the constitutive equations can be expressed as generalized Hooke's Law as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} + \begin{Bmatrix} \alpha_1 T \\ \alpha_2 T \\ 0 \end{Bmatrix} \tag{4}$$

where α_1 and α_2 are the coefficients of thermal expansion and T represents the temperature difference distribution. The relations defined by Equations (4) are valid when the axes of the material properties of the continuum are coincident with the axes selected for the differential equations of the problem. If the axes are not coincident, except for the z-axes, and the other two axes of the material properties are rotated about the z-axis through some angle, ϕ, in relation to the geometric axes, then more complicated relations between the stresses, temperature and strains are developed. Lekhnitskii [12] presents these relationships in some detail. For the generalized plane stress case in point the material coefficients are related to the two axis system by

$$\begin{bmatrix} a'_{11} \\ a'_{12} \\ a'_{21} \\ a'_{22} \\ a'_{66} \end{bmatrix} = \begin{bmatrix} m^4 & m^2n^2 & m^2n^2 & n^4 & 4m^2n^2 \\ m^2n^2 & m^4 & n^4 & m^2n^2 & -4m^2n^2 \\ m^2n^2 & n^4 & m^4 & m^2n^2 & -4m^2n^2 \\ n^4 & m^2n^2 & m^2n^2 & m^4 & 4m^2n^2 \\ m^2n^2 & -m^2n^2 & -m^2n^2 & m^2n^2 & (m^2-n^2) \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ a_{66} \end{bmatrix} \tag{5}$$

where $m = \cos\phi$, and $n = \sin\phi$.

No generality will be lost by continuing with the constitutive equation as given by Equations (4). Substituting Equations (4) into (3),

assuming that the a_{ij}'s are spatially dependent and carrying out the required differentiation yields:

$$\Big\{a_{12} \frac{\partial^2 \sigma_y}{\partial y^2} + a_{66} \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + a_{11} \frac{\partial^2 \sigma_x}{\partial y^2} + a_{21} \frac{\partial^2 \sigma_x}{\partial x^2} + a_{22} \frac{\partial^2 \sigma_y}{\partial x^2}$$

$$+ \frac{\partial^2}{\partial y^2} (\alpha_1 T) + \frac{\partial^2}{\partial x^2} (\alpha_2 T)\Big\}$$

$$+ \Big\{\frac{\partial^2 a_{11}}{\partial y^2} \sigma_x + \frac{\partial^2 a_{12}}{\partial y^2} \sigma_y + \frac{\partial^2 a_{21}}{\partial x^2} \sigma_x + \frac{\partial^2 a_{22}}{\partial x^2} \sigma_y$$

$$+ \frac{\partial^2 a_{66}}{\partial x \partial y} \sigma_{xy} + 2 \frac{\partial^2 a_{11}}{\partial y} \cdot \frac{\partial \sigma_x}{\partial y} + 2 \frac{\partial a_{12}}{\partial y} \cdot \frac{\partial \sigma_y}{\partial y} + 2 \frac{\partial a_{21}}{\partial x} \cdot \frac{\partial \sigma_x}{\partial x}$$

$$+ \frac{\partial a_{22}}{\partial x} \cdot \frac{\partial \sigma_y}{\partial x} + \frac{\partial a_{66}}{\partial x} \cdot \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial a_{66}}{\partial y} \cdot \frac{\partial \sigma_{xy}}{\partial x}\Big\} = 0 \,. \tag{6}$$

Assume that the body forces have a potential, V, such that

$$X = - \frac{\partial V}{\partial x}$$
$$Y = - \frac{\partial V}{\partial y} \,, \tag{7}$$

and choose Airy's stress function, Ψ, in the form

$$\sigma_x = \frac{\partial^2}{\partial y^2} + V$$
$$\sigma_y = \frac{\partial^2}{\partial x^2} + V \tag{8}$$
$$\sigma_{xy} = - \frac{\partial^2}{\partial x \partial y} \,.$$

Making the appropriate substitutions into Equation (6) yields

$$\left\{a_{22}\frac{\partial^4\psi}{\partial x^4} + (a_{21} + a_{11} - a_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2} + a_{11}\frac{\partial^4\psi}{\partial y^4}\right.$$

$$+ (a_{11} + a_{12})\frac{\partial^2 V}{\partial y^2} + (a_{22} + a_{21})\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2}{\partial y^2}(\alpha_1 T) + \left.\frac{\partial^2}{\partial x^2}(\alpha_2 T)\right\}$$

$$+ \left\{\left(\frac{\partial^2 a_{11}}{\partial y^2} + \frac{\partial^2 a_{21}}{\partial x^2}\right)\cdot\frac{\partial^2\psi}{\partial y^2} + \left(\frac{\partial^2 a_{12}}{\partial y^2} + \frac{\partial^2 a_{22}}{\partial x^2}\right)\cdot\frac{\partial^2\psi}{\partial x^2}\right.$$

$$- \frac{\partial^2 a_{66}}{\partial x\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2}{\partial x^2}(a_{22} + a_{21})\cdot V + \frac{\partial^2}{\partial y^2}(a_{11} + a_{12})\cdot V$$

$$+ \frac{\partial}{\partial y}(2a_{12} - a_{66})\cdot\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial}{\partial x}(2a_{21} - a_{66})\cdot\frac{\partial^3\psi}{\partial x\partial y} + 2\frac{\partial a_{11}}{\partial y}\cdot\frac{\partial^3\psi}{\partial y^3}$$

$$+ 2\frac{\partial a_{22}}{\partial x}\cdot\frac{\partial^3\psi}{\partial x^3} + \frac{\partial}{\partial y}(2a_{11} + 2a_{12})\cdot\frac{\partial V}{\partial y} + \frac{\partial}{\partial x}(2a_{21} + 2a_{22})\cdot\left.\frac{\partial V}{\partial x}\right\} = 0 \qquad (9)$$

and, of course, the equilibrium equations are met exactly.

Notice that in Equations (4), (5), and (9), a_{12} has not been equated to a_{21}. Normally, under the limit of small displacement theory dealing with linearly elastic materials which are conservative, the material coefficient matrix, defined in Equation (4), would be symmetric and a_{12} would equal a_{21}. However, some recent work by Bert and Guess [13], among others, shows that there exists experimentally derived data which indicate that for some types of composite materials exhibiting orthotropic properties the material coefficient matrix is not symmetric and a_{12} is not equal to a_{21}. To limit the growing complexity of this work, the material coefficient matrix will be taken as symmetric, at least for the initial phase of this effort.

Equation (9) can also be expressed in polar coordinates as follows:

$$\left\{ \left[a_{22} \frac{\partial^4\Psi}{\partial r^4} + 2 \frac{a_{22}}{r} \frac{\partial^3\Psi}{\partial r^3} - \frac{a_{11}}{r^2} \frac{\partial^2\Psi}{\partial r^2} + \frac{a_{11}}{r^3} \frac{\partial\Psi}{\partial r} + \frac{a_{11}}{r^4} \frac{\partial^4\Psi}{\partial\theta^4} \right.\right.$$

$$- \left(\frac{2a_{12} + a_{66}}{r^3}\right) \cdot \frac{\partial^3\Psi}{\partial r \partial\theta^2} + \left(\frac{2a_{12} + a_{66}}{r^2}\right) \cdot \frac{\partial^4\Psi}{\partial r^2 \partial\theta^2}$$

$$\left. + \left(\frac{2a_{11} + 2a_{12} + a_{66}}{r^4}\right) \frac{\partial^2\Psi}{\partial\theta^2}\right]$$

$$+ \left[a_{21} \frac{\partial^2}{\partial r^2} + \left(\frac{2a_{21} - a_{11}}{r}\right) \frac{\partial}{\partial r} + \frac{a_{11}}{r^2} \frac{\partial^2}{\partial\theta^2}\right] \left(V_r\right)$$

$$+ \left[a_{22} \frac{\partial^2}{\partial r^2} + \left(\frac{2a_{22} - a_{12}}{r}\right) \frac{\partial}{\partial r} + \frac{a_{12}}{r^2} \frac{\partial^2}{\partial\theta^2}\right] \left(V_\theta\right)$$

$$\left. + \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}\right]\left(\alpha_\theta T\right) + \left[\frac{1}{r^2} \frac{\partial^2}{\partial\theta^2} - \frac{1}{r} \frac{\partial}{\partial r}\right] \left(\alpha_r T\right) \right\}$$

$$+ \left\{\frac{\partial a_{12}}{\partial r} \cdot \left[\frac{1}{r} \frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r^2} \frac{\partial^3\Psi}{\partial r \partial\theta^2} - \frac{2}{r^3} \frac{\partial^2\Psi}{\partial\theta^2} + \frac{2\partial V_r}{\partial r} + \frac{2V_r - V_\theta}{r}\right]\right.$$

$$+ \frac{\partial a_{12}}{\partial\theta} \left[\frac{2}{r^2} \frac{\partial^3\Psi}{\partial r^2 \partial\theta} + \frac{2}{r^3} \frac{\partial V_\theta}{\partial\theta}\right] + \frac{\partial a_{22}}{\partial r} \left[2 \frac{\partial^3\Psi}{\partial r^3} + \frac{2}{r} \frac{\partial^2\Psi}{\partial r^2} + 2 \frac{\partial V_\theta}{\partial r} + \frac{2V_\theta}{r}\right]$$

$$- \frac{\partial a_{11}}{\partial r} \left[\frac{1}{r^3} \frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r^2} \frac{\partial\Psi}{\partial r} + \frac{V_r}{r}\right] + \frac{\partial a_{11}}{\partial\theta} \left[\frac{2}{r^4} \frac{\partial^3\Psi}{\partial\theta^3} + \frac{2}{r^3} \frac{\partial^2\Psi}{\partial r \partial\theta} + \frac{2}{r^2} \frac{\partial V_r}{\partial\theta}\right]$$

$$- \frac{\partial a_{66}}{\partial r} \left[\frac{1}{r^3} \frac{\partial^2\Psi}{\partial\theta^2} - \frac{1}{r^2} \frac{\partial^3\Psi}{\partial r \partial\theta^2}\right] + \frac{\partial a_{66}}{\partial\theta} \left[\frac{1}{r^4} \frac{\partial\Psi}{\partial\theta} - \frac{1}{r^3} \frac{\partial^2\Psi}{\partial r \partial\theta} + \frac{1}{r^2} \frac{\partial^3\Psi}{\partial r^2 \partial\theta}\right]$$

$$+ \frac{\partial^2 a_{12}}{\partial r^2} \left[\frac{1}{r^2} \frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r} \frac{\partial\Psi}{\partial r} + V_r\right] + \frac{\partial^2 a_{12}}{\partial\theta^2} \left[\frac{1}{r^2} \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r} V_\theta\right]$$

$$+ \frac{\partial^2 a_{22}}{\partial r^2} \left[\frac{\partial^2\Psi}{\partial r^2} + V_\theta\right] + \frac{\partial^2 a_{11}}{\partial\theta^2} \left[\frac{1}{r^4} \frac{\partial^2\Psi}{\partial\theta^2} + \frac{1}{r^3} \frac{\partial\Psi}{\partial r} + \frac{1}{r^2} V_r\right]$$

$$\left. + \frac{\partial^2 a_{66}}{\partial r \partial\theta} \left[\frac{1}{r^2} \frac{\partial^2\Psi}{\partial r \partial\theta} - \frac{1}{r^3} \frac{\partial\Psi}{\partial\theta}\right] \right\} = 0 \qquad (10)$$

with Ψ being the stress function defined by

$$\sigma_r = \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + V_r$$

$$\sigma_\theta = \frac{\partial^2\Psi}{\partial r^2} + V_\theta \tag{11}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right) .$$

Equations (11) satisfy the equilibrium equations formulated in polar coordinates for plane stress which are

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} + \Theta = 0 . \tag{12}$$

In order for Equations (11) to meet Equation (12) exactly the body functions V_r and V_θ must be defined as

$$V_\theta - \frac{\partial}{\partial r}(rV_r) = R$$

$$-\frac{\partial V_\theta}{\partial\theta} = \Theta , \tag{13}$$

thus putting a rather narrow interpretation on the body forces.

In the classical approach to the problem of orthotropic plane elasticity, the coefficients a_{ij} are either constants, as in the case of the homogeneous condition, or as in some rare cases, are special functions of position. In the first case all the terms within the second set of large braces, $\{\ \}$, in Equations (9) and (10) are zero leaving the remainder of these two equations in the form of the well-known, homogeneous, orthotropic,

compatibility equations in terms of the stress function Ψ. In the second case all the partial derivatives of the a_{ij}'s which appear in these second sets of braces are capable of being evaluated, resulting in extremely complicated fourth order partial differential equations with variable coefficients. Proceeding along classical lines, these equations must be solved for Ψ which contains arbitrary constants of integration. These constants are then determined by evaluating Ψ in terms of the stresses on the boundaries.

Suppose it is assumed that the material coefficients, the a_{ij}'s, are unknown but that the stress function Ψ is a fully defined function of the spatial coordinates. That is, the stresses throughout the body as well as on the boundary are known. In such a case, Equations (9) and (10) reduce to second order partial differential equations with variable coefficients, in terms of the a_{ij}'s. The solution of these equations and the resultant determination of the magnitude and distribution of the material properties throughout the body is defined as design synthesis.

It is believed that Equations (9) and (10) have not been previously published. Bert [14] derived an equation similar to (10) wherein he reduced the unknown material property coefficients from four to one. His formulation is as follows:

$$\begin{aligned} & S\left[\frac{\partial^4\Psi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\Psi}{\partial r^3} - \frac{e}{r^2}\frac{\partial^2\phi}{\partial r^2} + \frac{e}{r^3}\frac{\partial\Psi}{\partial r} + \frac{2(c-\nu)}{r^2}\frac{\partial^4\Psi}{\partial r^2\partial\theta^2}\right. \\ & \left. - \frac{2(c-\nu)}{r^3}\frac{\partial^3\Psi}{\partial r\partial\theta^2} + \frac{2(c-\nu+e)}{r^4}\frac{\partial^2\phi}{\partial\theta^2} + \frac{e}{\gamma^4}\frac{\partial^4\Psi}{\partial\theta^4}\right] \\ & + \frac{dS}{dr}\left[2\frac{\partial^3\Psi}{\partial r^3} + \frac{2-\nu}{r}\frac{\partial^2\Psi}{\partial r^2} - \frac{e}{r^2}\frac{\partial\Psi}{\partial r} + \frac{2(c-\nu)}{r^2}\frac{\partial^3\Psi}{\partial r\partial\theta^2}\right. \\ & \left. - \frac{2(c-\nu)+e}{r^3}\frac{\partial^2\Psi}{\partial\theta^2}\right] + \frac{d^2S}{dr^2}\left[\frac{\partial^2\Psi}{\partial r^2} - \frac{\nu}{r}\frac{\partial\Psi}{\partial r} - \frac{\nu}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right] \\ & - \left[\nu\frac{\partial^2}{\partial r^2} - \frac{e-2\nu}{r}\frac{\partial}{\partial r} - \frac{e}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left(SV_r\right) \\ & + \left[\frac{\partial^2}{\partial r^2} + \frac{2+\nu}{r}\frac{\partial}{\partial r} - \frac{\nu}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\left(SV_\theta\right) = 0\ , \end{aligned} \tag{14}$$

with the thermal terms omitted. Equation (14) is equal to Equation (10) with the following identities:

$$
\begin{aligned}
S &\equiv a_{22} \\
e &\equiv a_{11}/a_{22} \\
c &\equiv a_{66}/2a_{22} \\
\nu &\equiv - a_{12}/a_{22}
\end{aligned}
\qquad (15)
$$

with S being the dependent variable and e, c, and ν being fixed ratios. It is anticipated that most real composite materials will exhibit material property characteristics as defined by Equations (15).

The solution of equations such as (9) and (10) where there are 4 or more independent a_{ij}'s, even where these a_{ij}'s are assumed independent of temperature, is quite difficult but certainly not impossible. The simplifying assumption made in Equations (15) leading to the formulation of Equation (14) reduces the problem to only one unknown parameter. Work is progressing on the application of Equations (9), (10), and (14) to the concept of design synthesis. The following section of this paper deals with such applications.

APPLICATIONS

Rotationally Symmetric Problems

Example 1: The pressurized annular disk, internal pressure: Consider a pressurized annular disk as shown in Figure 1. In the case where the ring is isotropic and homogeneous, the stress distribution is as developed by Lamé (1852) and is given by (c.f., Ref. [15], p. 60),

$$\sigma_r = -\frac{P}{k^2 - 1}\left(\frac{1}{\rho^2} - 1\right)$$
$$\sigma_\theta = \frac{P}{k^2 - 1}\left(\frac{1}{\rho^2} + 1\right) \tag{16}$$

where $\rho = r/b$, and $k = b/a$. These relations lead to the following conclusions:

(1) $|\sigma_\theta| > |\sigma_r|$ for all ρ and all k

(2) $(\sigma_\theta)_{max}$ occurs at the inner boundary ($\rho = 1/k$), and thus

$$(\sigma_\theta)_{max} = P\left(\frac{k^2 + 1}{k^2 - 1}\right) > P \tag{17}$$

From the stresses so generated, which for the homogeneous ring, are quite independent of the material properties, it is clear that the material is not being used efficiently, particularly as the thickness ratio, k, increases.

For the homogeneous, orthotropic annular disk Bienick et al. [16] shows that the stresses are given by

$$\sigma_r = c_1 r^{-(f+1)} + c_2 r^{f-1}$$
$$\sigma_\theta = -c_1 f r^{-(f+1)} + c_2 f r^{f-1} \tag{18}$$

where $f = (a_{11}/a_{22})^{1/2}$, and c_1 and c_2 are determined from the boundary conditions, in this case $\sigma_r = -P$ at $r = a$ and $\sigma_r = 0$ at $r = b$. This work and

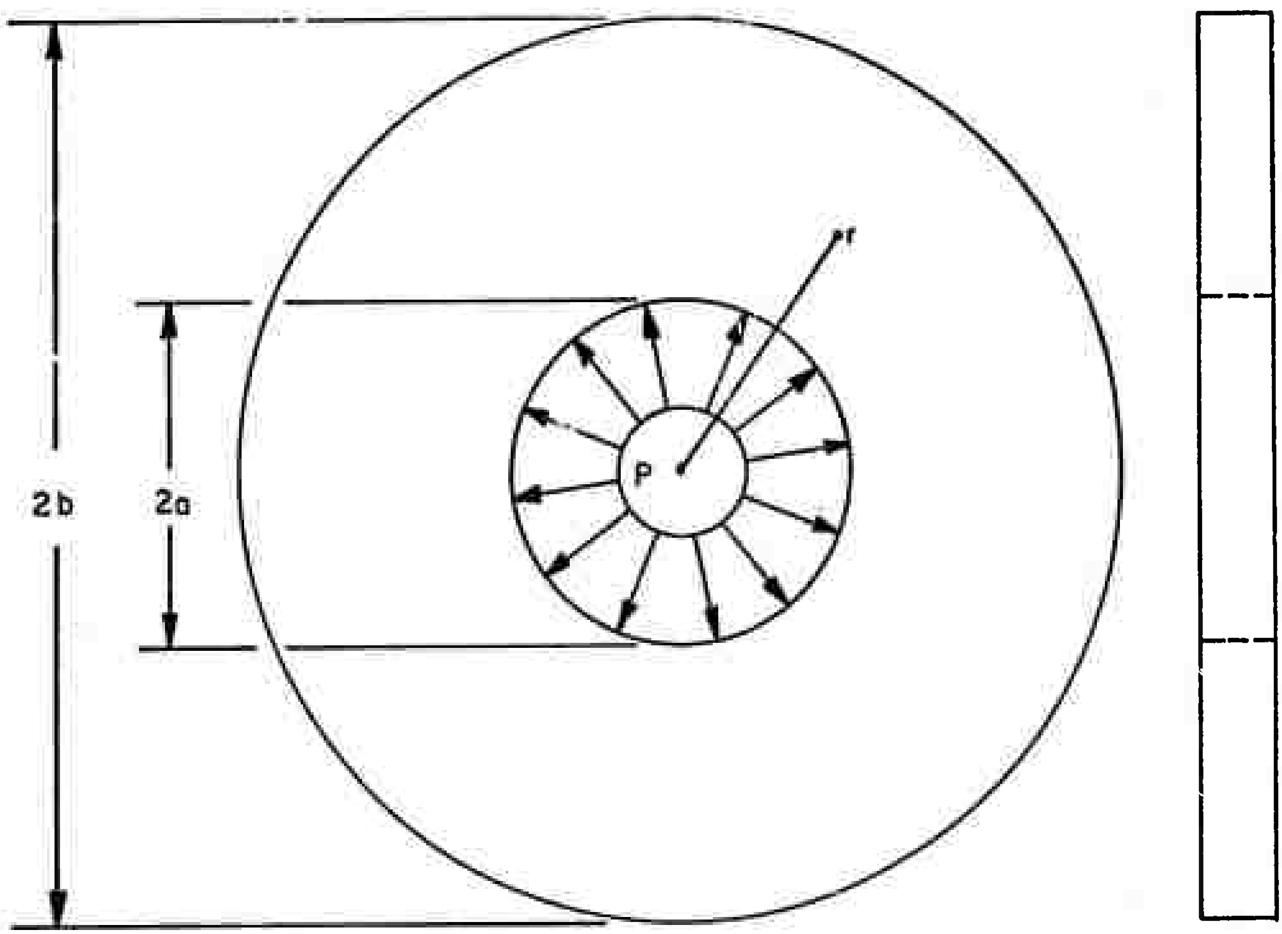

FIGURE 1. PRESSURIZED ANNULAR DISK

work done by Shaffer [17] show that as k gets large the limit of the stress concentration approaches the orthotropy ratio, f, and does so rapidly. Both Bienick and Shaffer deal with the nonhomogeneous case, but both assume a special form of nonhomogeneity, mainly that

$$a_{11} = \bar{a}_{11}\, r^{\lambda}$$

$$a_{22} = \bar{a}_{22}\, r^{\lambda}$$

$$a_{12} = a_{21} = \bar{a}_{12}\, r^{\lambda}$$

where λ is real and $\bar{a}_{11}$, $\bar{a}_{22}$, and $\bar{a}_{12}$ are constants. Both carry out optimization by varying λ and observing the results, but no attempt was made to carry out design synthesis directly.

Consider now that the disk is composed of a heterogeneous, orthotropic material where the material properties are undefined but are functions of the radius of the disk. The equilibrium equation for this case reduces to

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \,. \tag{19}$$

The compatibility equation reduces to

$$\frac{d}{dr}\,(r\epsilon_\theta) - \epsilon_r = 0 \,, \tag{20}$$

and the stress strain relation for an orthotropic medium is

$$\begin{aligned} \epsilon_r &= a_{11}\sigma_r + a_{12}\sigma_\theta \\ \epsilon_\theta &= a_{12}\sigma_r + a_{22}\sigma_\theta \end{aligned} \tag{21}$$

where a_{11}, a_{12}, and a_{22} are functions of r. Assuming a stress function such that

$$\sigma_r = \frac{1}{r}\Psi$$
$$\sigma_\theta = \frac{d\Psi}{dr} \tag{22}$$

and substituting Equations (21) and (22) into Equation (20) and carrying out the required differentiation results in

$$a_{22}\Psi'' + (a'_{22} + \frac{1}{r}a_{22})\Psi' + (\frac{1}{r}a'_{12} - \frac{1}{r^2}a_{11})\Psi = 0 , \tag{23}$$

where the prime marks designate absolute derivatives with respect to r.

Equation (23) is a single differential equation with three independent material coefficients as the operational parameters. When dealing with specific materials they may be found to be completely independent or show some type of defined relationship. As a first step we will assume that there does exist a simple ratio relationship between them that is expressable as

$$a_{22} = a_{22} \quad , \quad a'_{22} = a'_{22}$$
$$a_{11} = k_1 a_{22}, \quad a'_{11} = k_1 a'_{22} \tag{24}$$
$$a_{12} = k_2 a_{22}, \quad a'_{12} = k_2 a'_{22} \ .$$

Making the appropriate substitutions in Equation (23) yields

$$\left[\Psi' + \frac{k_2}{r}\Psi\right] a'_{22} + \left[\Psi'' + \frac{1}{r}\Psi' - \frac{k_1}{r^2}\Psi\right] a_{22} = 0 \ . \tag{25}$$

Thus if Ψ is known a_{22} is fully defined. Let us suppose that for effective material utilization it is desired that σ_θ be constant throughout the cylinder; i.e., $\sigma_\theta = A_o$. Integrating the second of Equations (22) gives

$$\Psi = A_o r + B_o \ . \tag{26}$$

Applying the boundary conditions that $\sigma_r(a) = - P$ and $\sigma_r(b) = 0$ yields

$$\Psi = \frac{Pb}{k - 1} [\rho - 1]$$

$$\sigma_r = - \frac{P}{k - 1} [\frac{1}{\rho} - 1] \tag{27}$$

$$\sigma_\theta = \frac{P}{k - 1}$$

with $\rho = r/b$ and $k = b/a$ as before. It is interesting to compare Equations (27) with Equations (16) and (17). From Equations (27) it can be seen that if k is greater than 2, σ_θ becomes less than the pressure P, and σ_r, which equals P at $\rho = 1/k$, becomes the maximum normal stress in absolute value. Thus for such a stress function and geometry there is no effective stress concentration. This is, of course, never true for the homogeneous disk.

With the stress function now fully defined, Equation (25) becomes

$$\left[(1+k_2) - \frac{k_2 b}{r}\right] a'_{22} + \left[\frac{(1-k_1)}{r} + \frac{k_1 b}{r^2}\right] a_{22} = 0 \, . \tag{28}$$

which has the solution

$$a_{22} = C_o(\rho)^{\beta-\xi} \left[\frac{k_2}{\rho} + (1-k_2)\right]^{-\xi} \tag{29}$$

where $\rho = r/b$

$\beta = - k_1/k_2$

$\xi = - \left[\frac{k_1 - k_2}{(1-k_2)k_2}\right]$,

and for the modified orthotropic condition as defined by Equations (21) and (24)

$$a_{22} = a_{22} = \frac{1}{E_\theta}$$

$$k_1 = \frac{a_{11}}{a_{22}} = \frac{E_\theta}{E_r} \tag{30}$$

$$k_2 = -\frac{a_{12}}{a_{22}} = \nu_{\theta r} \, .$$

The inverse of Equation (29) or $E_\theta(r)$ is shown plotted in Figure 2 for various k_1's while k_2 was held constant at 0.5. This figure shows the strong dependency of the modulus distribution upon the orthotropy ratio k_1. The ratio k_2 has a lesser effect as shown by Figure 3. Here the isotropic case ($k_1 = 1$) is shown plotted for four values of k_2.

Example 2. Pressurized annular disk, external pressure: Consider the pressurized annular disk as shown in Figure 1 but with the pressure acting on the OD rather than the ID as shown. If the stress criterion is retained; i.e., σ_θ = constant, then the stress function becomes

$$\Psi = \frac{qb}{k-1}[1 - \rho k]$$

$$\sigma_r = \frac{q}{k-1}[\frac{1}{\rho} - k] \qquad (31)$$

$$\sigma_\theta = -\frac{qk}{k-1}$$

where q = external pressure

$k = b/a$

$\rho = r/b$.

The compatibility equation (25) becomes

$$\left[(1+k_2) - \frac{k_2 a}{r}\right] a'_{22} + \left[\frac{(1-k_1)}{r} + \frac{k_1 a}{r^2}\right] a_{22} = 0 . \qquad (32)$$

Equation (32) is the same as Equation (28) except that a replaces b. The solution of Equation (32) is

$$a_{22} = C_o(\rho)^{\beta-\xi}\left[\frac{k_2}{\rho k} + (1-k_2)\right]^{-\xi} \qquad (33)$$

where β, ξ, and ρ are defined in Equation (29). Comparing Equations (33) and (29) we see that they are not the same, differing by the quantity 1/k

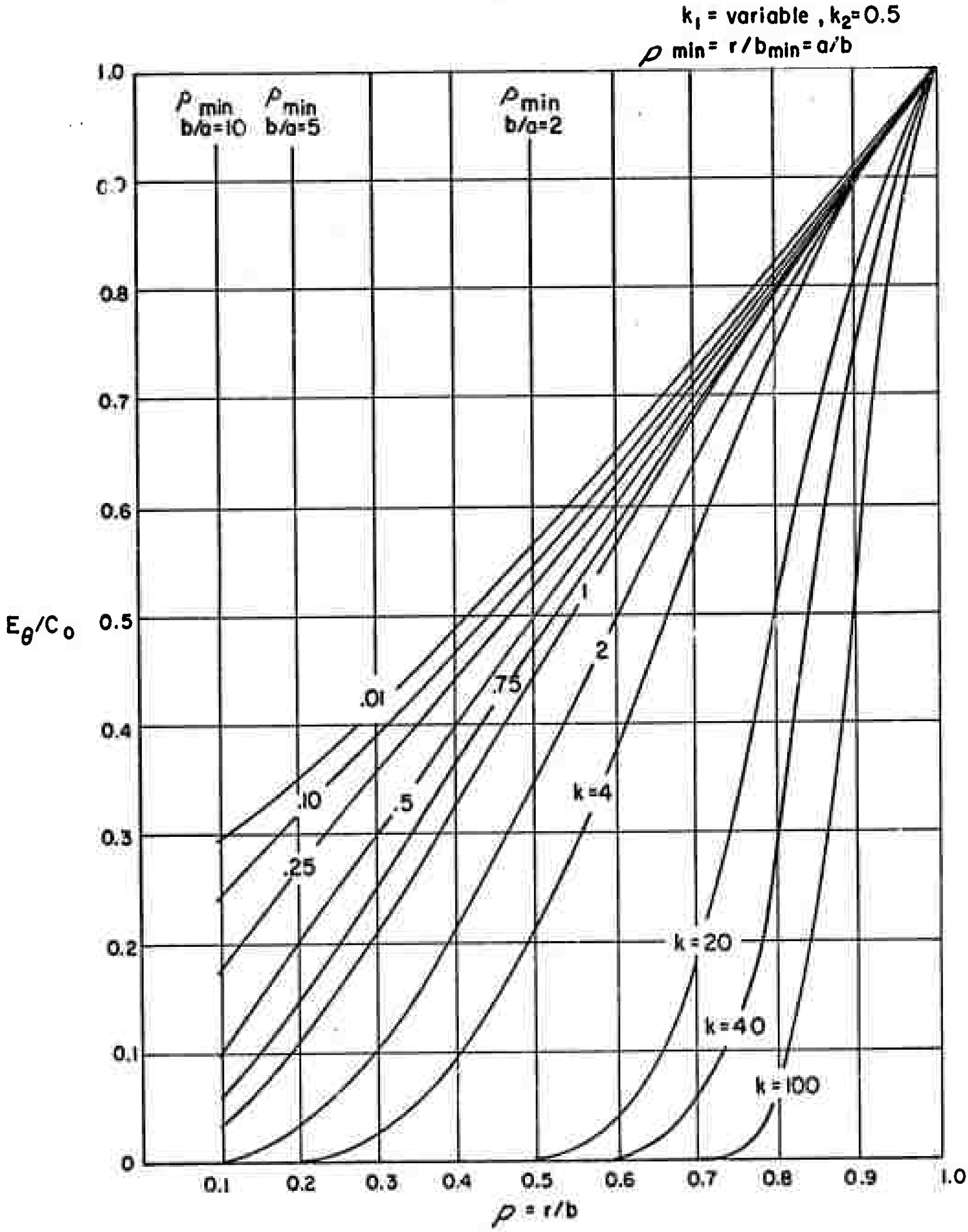

FIGURE 2. MATERIAL PROPERTY VARIATION FOR PRESSURIZED DISK WITH σ_θ = CONSTANT, INTERNAL PRESSURE

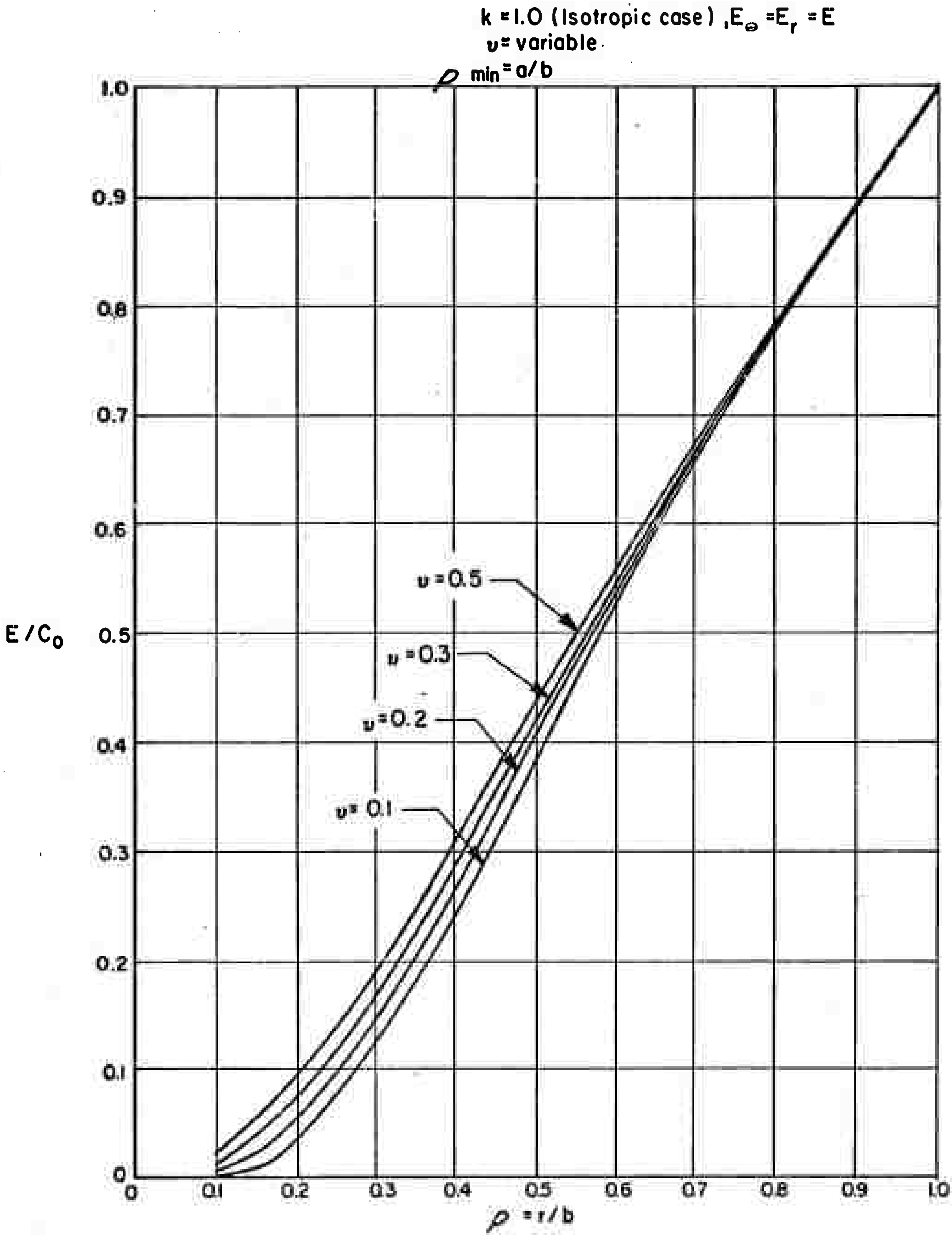

FIGURE 3. MATERIAL PROPERTY VARIATION FOR PRESSURIZED ANNULAR DISK WITH σ_θ = CONSTANT, INTERNAL PRESSURE (ISOTROPIC CASE)

within the second factor. Thus the distribution of the material parameters through the thickness of the disk must be different as compared to the internally pressurized disk. This is shown in Figures 4 and 5. Also notice that for the internally pressurized disk the hoop stress σ_θ tends to zero as k tends to infinity while the limit on σ_θ for the externally pressurized disk is q, the pressure. Thus for the externally pressurized case the material is not being as effectively utilized and perhaps some other criterion might more suitably apply.

If the annulus has both internal and external pressure and the same stress criterion is applied, i.e., σ_θ = constant, then

$$\begin{aligned} \Psi &= \frac{Pb}{k-1}\left[\rho - 1\right] - \frac{qb}{k-1}\left[1 - \rho k\right] \\ \sigma_r &= -\frac{P}{k-1}\left[\frac{1}{\rho} - 1\right] - \frac{q}{k-1}\left[\frac{1}{\rho} - k\right] \qquad (34) \\ \sigma_\theta &= \frac{1}{k-1}\left[P - qk\right] , \end{aligned}$$

and the material property variation is given by

$$a_{22} = C_o(\rho)^{\beta-\xi}\left[\frac{k_2}{\rho} + (1-k_2)\left(\frac{P_a - qb}{P_a - qa}\right)\right]^{-\xi} \qquad (35)$$

Example 3: The rotating disk: Consider the rotating, uniform thickness annular disk as shown in Figure 6. The equilibrium equation governing this case is

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \frac{\gamma}{g}\omega^2 r^2 = 0 \, . \qquad (36)$$

The compatibility equation in terms of strain and the stress-strain relations for an orthotropic material are given by Equations (20) and (21), respectively. Substituting Equation (21) into (20) results in

$$0 = a'_{12}\sigma_r + a_{12}\sigma'_r + a'_{22}\sigma_\theta + a_{22}\sigma'_\theta + \frac{1}{r}\left[(a_{12}-a_{11})\sigma_r + (a_{22}-a_{12})\sigma_\theta\right]. \qquad (37)$$

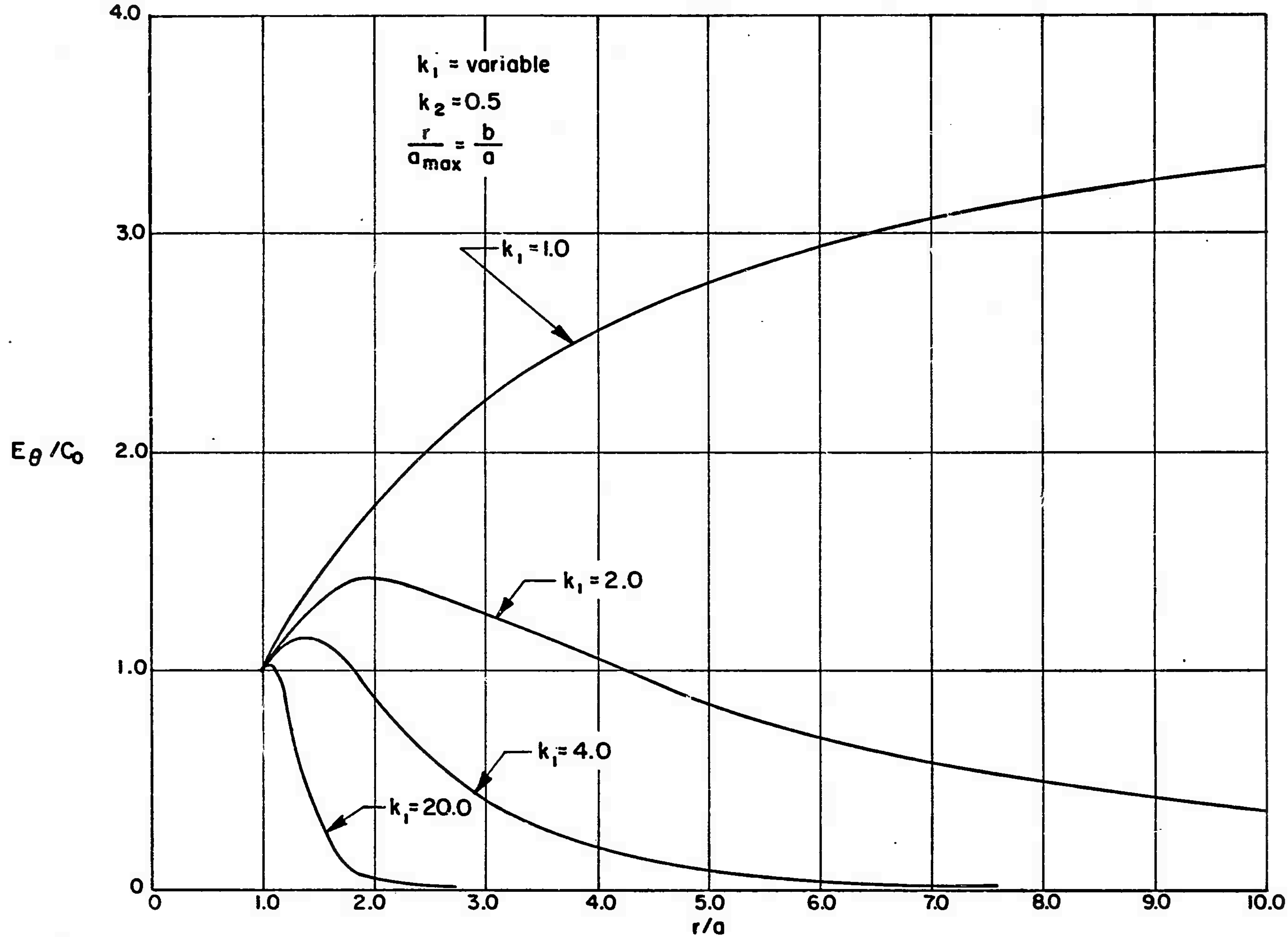

FIGURE 4. MATERIAL PROPERTY VARIATION FOR PRESSURIZED ANNULAR DISK FOR σ_θ = CONSTANT, EXTERNAL PRESSURE

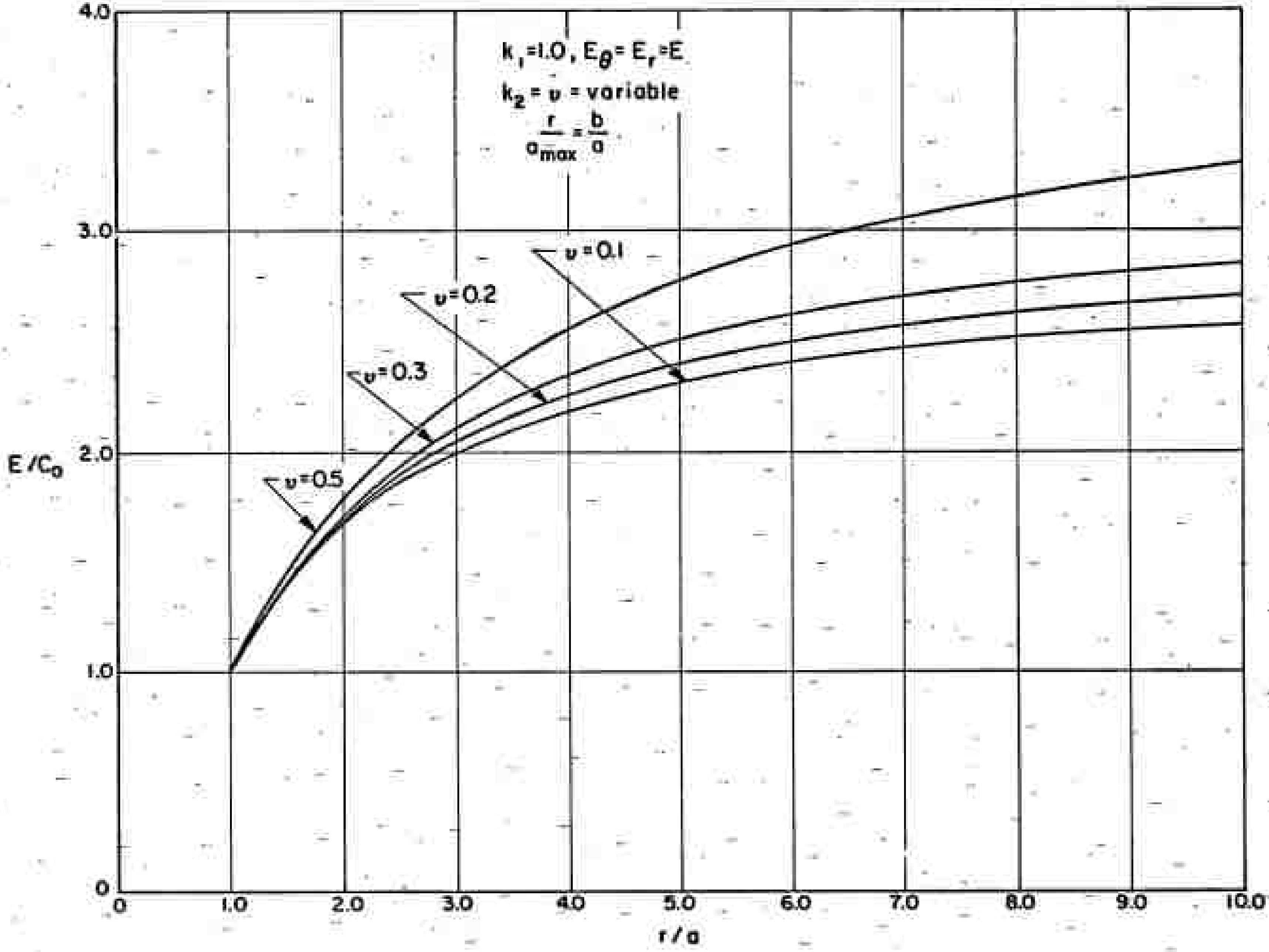

FIGURE 5. MATERIAL PROPERTY VARIATION FOR PRESSURIZED ANNULAR DISK, FOR σ_0 = CONSTANT, EXTERNAL PRESSURE (ISOTROPIC CASE)

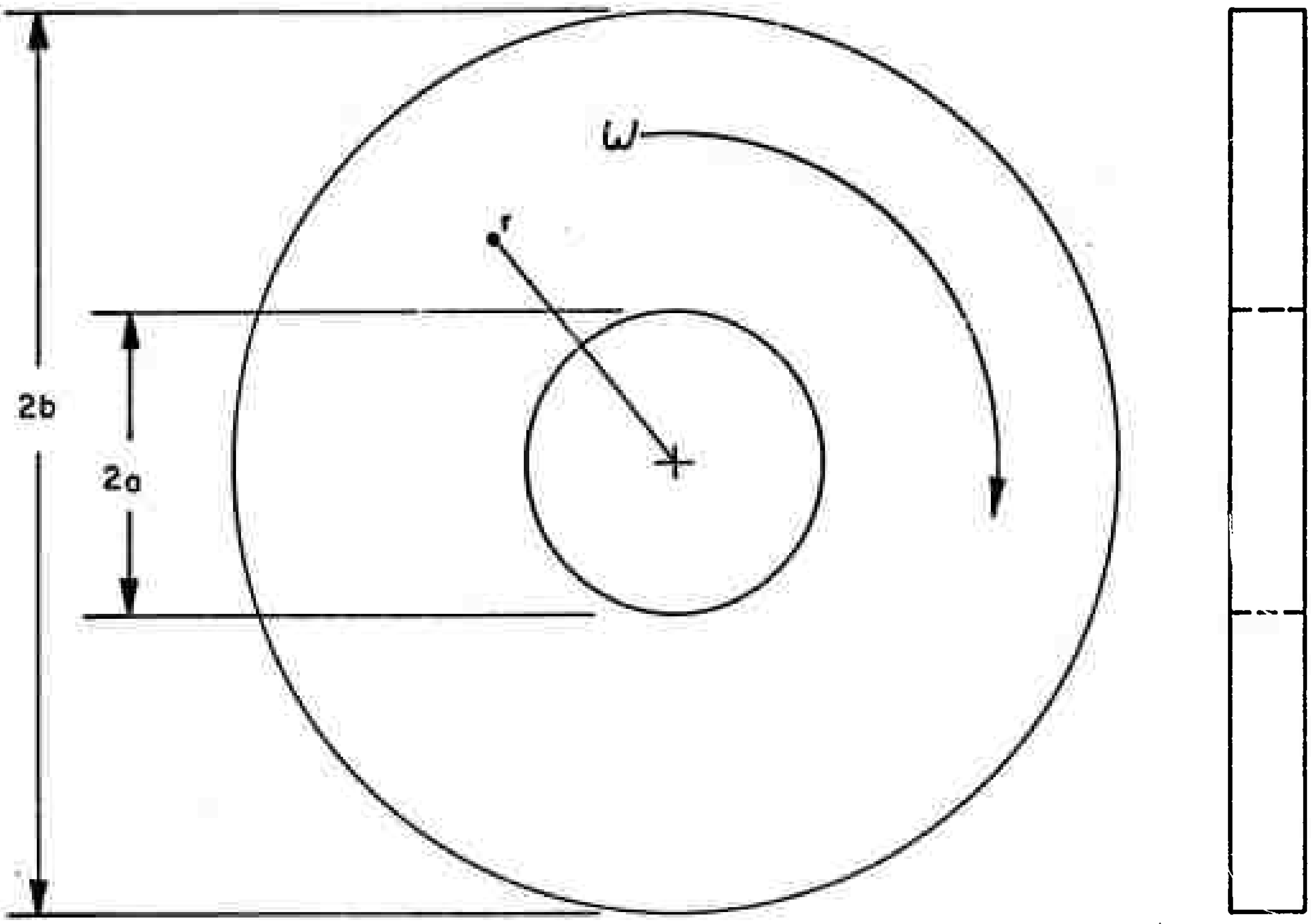

FIGURE 6. ROTATING ANNULAR DISK

Assume the stress function

$$\sigma_r = \frac{1}{r}\Psi$$
$$\sigma_\theta = \Psi' + \frac{\gamma}{g}\omega^2 r^2 . \tag{38}$$

It is possible that the material density, γ, could be a function of the radius. At this stage such a condition introduces an unnecessary complication and will not be considered. This being the case the governing compatibility equation becomes

$$\Psi''(a_{22}) + \Psi'(a'_{22} + \frac{1}{r}a_{22}) + \Psi\left(\frac{a'_{12}}{r} - \frac{a_{11}}{r^2}\right)$$
$$+ \frac{\gamma}{g}\omega^2 r^2\left(a'_{22} + 3\frac{a_{22}}{r} - \frac{a_{12}}{r}\right) = 0 . \tag{39}$$

Again, for simplicity, the modified orthotropy relations as defined by Equations (24) will be adapted making Equation (39) take the form

$$0 = a'_{22}\left[\Psi' + \frac{k_2}{r}\Psi + \frac{\gamma}{gg}\omega^2 r^2\right] + a_{22}\left[\Psi'' + \frac{\Psi'}{r} - \frac{k_1\Psi}{r^2} + \frac{\gamma}{g}\omega^2 r(3-k_2)\right] \tag{40}$$

The stress criterion will be the same as before, i.e., σ_θ = constant. Using this criterion and operating on the second of Equations (38) together with the assumed boundary conditions that $\sigma_r(a) = \sigma_r(b) = 0$ yields

$$\Psi = \frac{\gamma v^2}{3g}\left(\frac{k}{k-1}\right)\left[r\left(1 - \frac{1}{k^3}\right) - a\left(1 - \frac{1}{k^2}\right) - \left(\frac{r}{b}\right)^3\left(b - a\right)\right]$$
$$\sigma_r = \frac{\gamma v^2}{3g}\left(\frac{k}{k-1}\right)\left[1 - \frac{1}{k^3} - \frac{1}{\rho k} + \frac{1}{\rho k^3} - \rho^2 + \frac{\rho^2}{k}\right] \tag{41}$$
$$\sigma_\theta = \frac{\gamma v^2}{3g}\left(\frac{k}{k-1}\right)\left[\frac{k^3 - 1}{k^3}\right]$$

where $\rho = r/b$

$k = b/a$

$v = \omega b$ = tip velocity

ω = rotational velocity, radians

g = acceleration of gravity.

Here we note that in the limits, when $k \to 1$

$$\sigma_\theta = \frac{\gamma v^2}{g} \tag{42}$$

which is the stress in a rotating thin ring, and when $k \to \infty$ (i.e., when very small)

$$\sigma_\theta = \frac{\gamma v^2}{3g} , \tag{43}$$

which is smaller than exists for the isotropic, homogeneous case by the ratio

$$\left(\frac{\sigma_{\theta_1}}{\sigma_{\theta_o}}\right)_{max} = \frac{4}{9+\nu} \tag{44}$$

where σ_{θ_1} = hoop stress for heterogeneous case

σ_{θ_o} = hoop stress for isotropic, homogeneous case

ν = Poisson's ratio for isotropic, homogeneous case .

Substituting Equation (41) into Equation (40) and carrying out the required differentiation yields

$$a'_{22} - \left[\frac{Ar^3 + Br + D}{Fr^4 + Gr^2 + Hr}\right] a_{22} = 0 \tag{45}$$

where $A = (k_1 - 3k_2)$

$B = (1 - k_1) \left[\frac{b^3 - a^3}{b - a}\right]$

$$D = k_1(ab)(b + a)$$

$$F = k_2$$

$$G = -(1 + k_2)\left[\frac{b^3 - a^3}{b - a}\right]$$

$$H = k_2(ab)(b + a)$$

Equation (45) is not readily solvable in closed form. Solutions are presently being developed utilizing a finite difference approximation technique.

Non-Symmetric Problems

Example 4: Small hole in an infinite plate. Figure 7 represents a small hole in an infinite plate which is subjected to a uniform tensile stress, P, in the x-direction. For the homogeneous, isotropic condition, the stress distribution around the hole is well known as given by Timoshenko [15] as

$$\sigma_r = \frac{P}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{P}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{P}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{P}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{46}$$

$$\sigma_{r\theta} = -\frac{P}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta \quad .$$

The maximum stress occurs at $r = a$, $\theta = (\pi/2, 3\pi/2)$, and is

$$\sigma_{max} = (\sigma_\theta)_{r=0,\theta=\pi/2} = 3P \ .$$

For the homogeneous, anisotropic condition work by Green and Zerna [18], Hearmon [19], Savin [20], Leckhniskii [12], and (as directly applied to composites) by Greszczuk [21], shows that the maximum stress at the hole is always greater than for the isotropic case and can reach values as high as 9P.

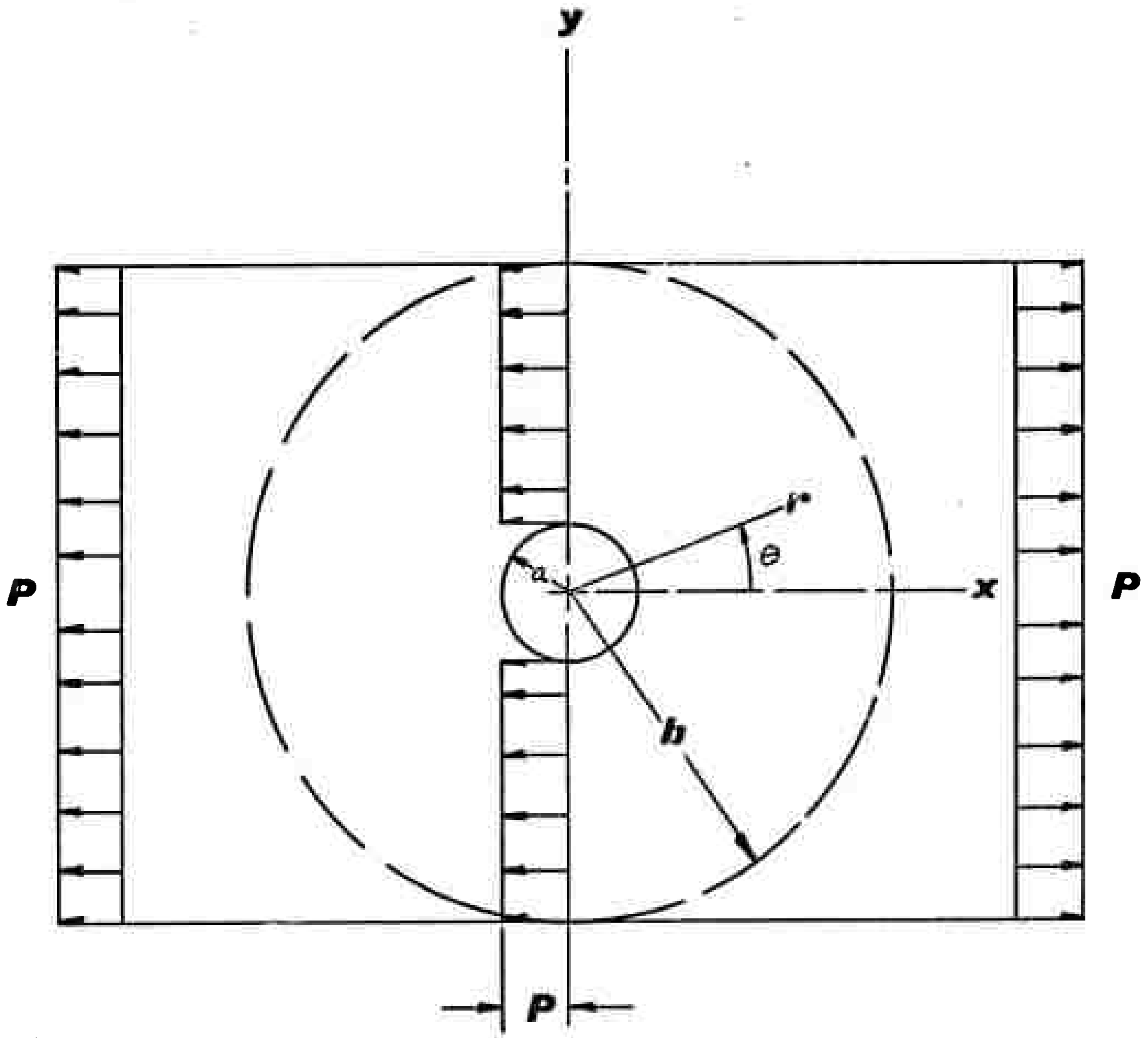

FIGURE 7. SMALL HOLE IN INFINITE PLATE SUBJECTED TO UNIFORM, UNIAXIAL TENSILE STRESS FIELD

Referring to Figure 7, consider the portion of the plate within a concentric circle of radius b, large in comparison with a. It can safely be assumed that the stresses at radius b are effectively the same as in a plate without the hole and can be given by

$$(\sigma_r)_{r=b} \approx \frac{1}{2} P (1 + \cos 2\theta)$$
$$(\sigma_{r\theta})_{r=b} \approx - \frac{1}{2} P \sin 2\theta \ . \tag{47}$$

From Equation (46) it can be seen that

$$(\sigma_\theta)_{r=b} \approx \frac{1}{2} P (1 - \cos 2\theta).$$

It seems reasonable, then, to choose a stress criterion for the plate

$$\sigma_\theta = \text{function of } \theta = \frac{1}{2} P (1 - \cos 2\theta). \tag{48}$$

From the second of Equations (11), with V equal to zero the stress function becomes

$$\Psi = \frac{1}{4} Pr^2 (1 - \cos 2\theta) + f_1(\theta) r + f_2(\theta) \ . \tag{49}$$

where $f_1(\theta)$ and $f_2(\theta)$ are functions of θ only. Applying the first and third of Equations (11) to Ψ results in

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) = - \sigma_{r\theta} = \frac{1}{2} P \sin 2\theta - \frac{1}{r^2} \cdot \frac{df_2(\theta)}{d\theta}$$
$$\frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = \sigma_r = \frac{1}{2} P (1 + \cos 2\theta) + \frac{1}{r}\frac{d^2 f_1(\theta)}{r d\theta^2} + \frac{1}{r^2}\frac{d^2 f_2(\theta)}{d\theta^2} + \frac{1}{r} f_1(\theta) \tag{50}$$

Applying the boundary conditions

$$(\sigma_r)_{r=a} = (\sigma_{r\theta})_{r=a} = 0$$

yields

$$f_2(\theta) = -\frac{Pa^2}{4}\cos 2\theta$$

$$\frac{d^2 f_1(\theta)}{d\theta^2} + f_1(\theta) = -\frac{P}{2}[1 + 3\cos 2\theta] \; . \qquad (51)$$

Choosing as a particular solution to the second of Equations (51)

$$f_1(\theta) = C_1 + C_2 \cos 2\theta \qquad (52)$$

yields

$$f_1(\theta) = -\frac{Pa}{2}[1 - \cos 2\theta]$$

and results in

$$\Psi = \frac{P}{4}\left\{(r^2 - 2ar) - (r^2 - a^2)\cos 2\theta\right\}$$

$$\sigma_r = \frac{P}{2}\left(1 - \frac{a}{r}\right)\left[1 - \cos 2\theta + 2\left(1 - \frac{a}{r}\right)\cos 2\theta\right] \qquad (53)$$

$$\sigma_\theta = \frac{P}{2}(1 - \cos 2\theta)$$

$$\sigma_{r\theta} = -\frac{P}{2}\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta$$

The stress function Ψ as defined by the first of Equations (53) has been applied to Equation (10) with the body functions and the temperature difference distribution taken as zero. Further, Equations (24) have been utilized to initially simplify the problem and numerical solutions are being developed.

SUMMARY

Mathematical design synthesis has been shown to be possible in certain specific applications. The selection of a design criterion, in the cases discussed, one dealing with stress distribution, and the development of the material distribution within a plane body such that compatibility is satisfied, appears to be a rational basis of design for composite materials. It is to be shown that this concept of design synthesis can be applied to the fabrication of real composite structures.

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